A Tutorial Introduction to TLAPS

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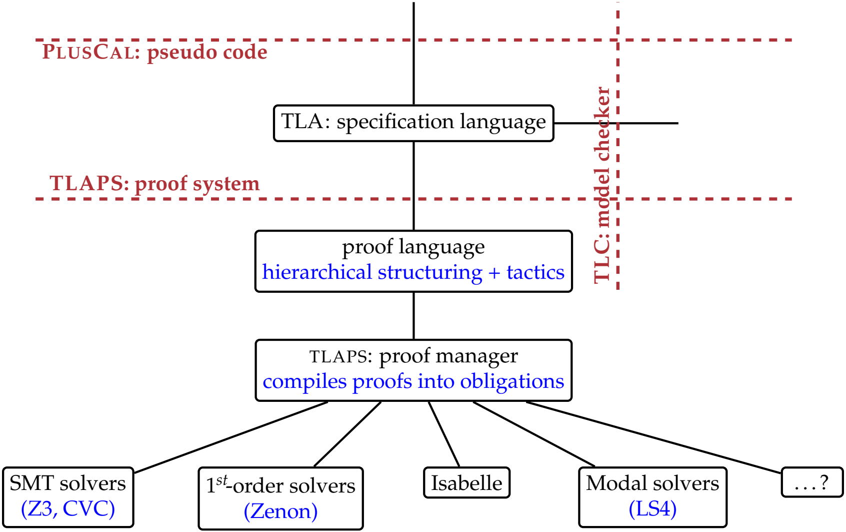
TLA+ Community Event, ABZ 2014
Toulouse, June 3, 2014

Outline

TLAPS in Context



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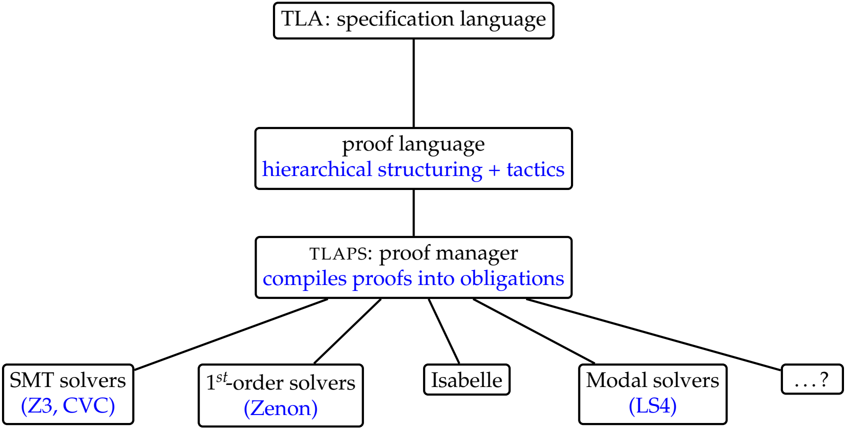
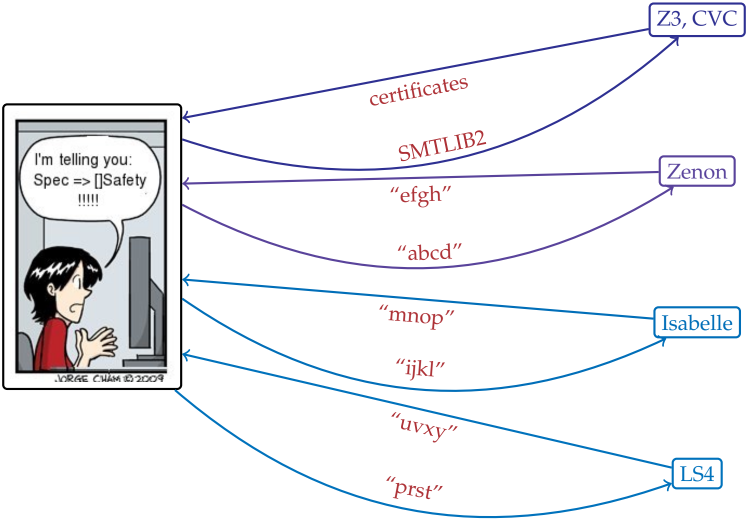


TLAPS in Context

TLAPS essentially does two things:

- translate between TLA and the languages that the backend provers understand;
- help the user break up a theorem P into obligations $O_1 \ldots O_n$, while maintaining the fact that $O_1 \land \ldots \land O_n \Rightarrow P$.

Talking to ATPs about TLA Specs



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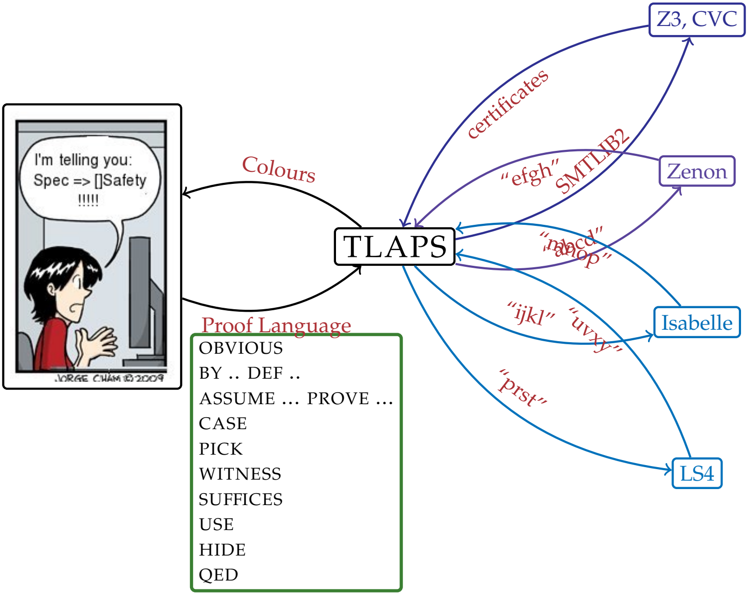
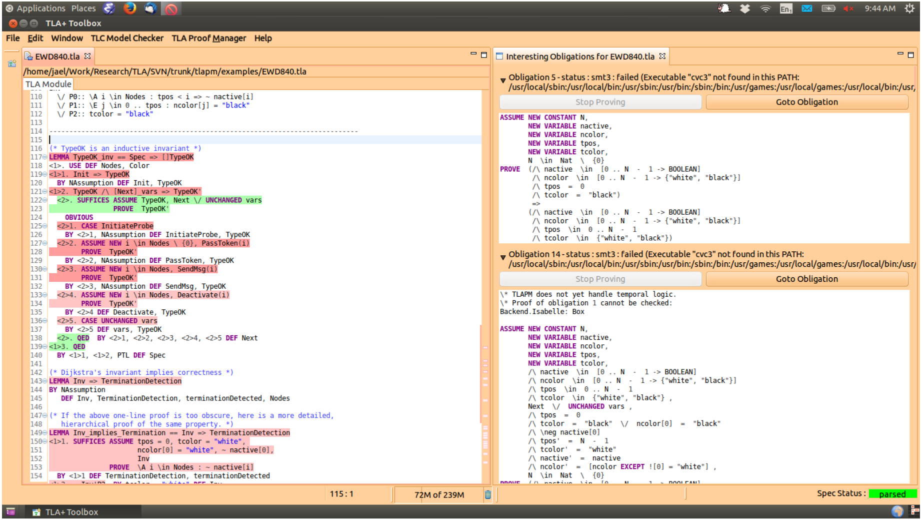


Hence the way the interface looks:



conversation

user $\Longleftrightarrow$ TLAPS

snipplets of conversations

TLAPS $\Longleftrightarrow$ backend provers

There are two kinds of TLAPS proofs:

one-liners

- OBVIOUS
- BY ... [DEF ...]

Each one-line proof generates one obligation.

(Or there abouts...)

hierarchical

```
⟨1⟩1 X
  ⟨2⟩1 Y
    OBVIOUS
    ⋮
  ⟨2⟩q QED
    BY ... DEF ...
⋮
⟨1⟩q QED
  BY ... DEF ...
```

Obligations

An *obligation* is a claim of the form $\Gamma \vdash P$,
which is translated and handed on to the backend provers.

Obligations

An *obligation* is a claim of the form ASSUME Γ PROVE P,
which is translated and handed on to the backend provers.

To prove an obligation, by default, TLAPS will ask:

1. CVC
2. Zenon
3. Isabelle

But one can change that...

An *obligation* is a claim of the form ASSUME Γ PROVE P,
which is translated and handed on to the backend provers.

The TLAPS game is mainly to construct obligations so that:

1. they are true, i.e.:
   1. Γ contains all relevant facts), and
   2. all relevant definitions are unfolded;
2. they are not too big for the backend provers to handle.

Obligations: Controlling Γ

Once one has one's logic right, the game is to control Γ.

By default:

- all constant-/variable-declarations, with domain-assumptions, are in Γ;
- no definitions are unfolded.

Once one has one's logic right, the game is to control Γ.

named & un-named steps

$\langle 1 \rangle 1 \;\; X$
$\quad \ldots$

$\langle 1 \rangle \;\; Y$
$\quad \ldots$

$\langle 1 \rangle 3 \;\; Z$
(* here Y is in Γ, but X is not *)
BY $\langle 1 \rangle 1$ (* here Y and X are in Γ *)

USE & HIDE

The keywords USE resp. HIDE include in resp. remove from Γ steps, theorems or assumptions;

USE [DEF] resp. HIDE [DEF] fold resp. unfold definitions in Γ.

Whether a USE- and HIDE-step is named or un-named does not matter.

Writing a simple Hierarchical Proof

quick recap: EWD 840

Writing a simple Hierarchical Proof

The safety-proof has the following structure:

LEMMA $Spec \Rightarrow \Box TerminationDetection$

(* Dijkstra's invariant implies correctness *)
$\langle 1 \rangle 1\ \ Inv \Rightarrow TerminationDetection$

(* Dijkstra's invariant is (trivially) established by the initial condition *)
$\langle 1 \rangle 2\ \ Init \Rightarrow Inv$

(* Dijkstra's invariant is inductive relative to the type invariant *)
$\langle 1 \rangle 3\ \ TypeOK \wedge Inv \wedge [Next]_{vars} \Rightarrow Inv'$

$\langle 1 \rangle q\ \ QED$
BY $\langle 1 \rangle 1$, $\langle 1 \rangle 2$, $\langle 1 \rangle 3$, $TypeOK_{inv}$, PTL DEF $Spec$

writing a simple hierarchical proof

Some more Proof Constructs: WITNESS

When proving a goal of the form:

$$\exists x \in S : \; P(x)$$

To prove it we can write:

$$\langle 1 \rangle 6 \;\; \textsc{witness} \;\; a \in S$$

for some a already in Γ.

The effect is:

1. step $\langle 1 \rangle 6$ needs a proof that $a \in S$;
2. the goal from now on is $P(a)$.

Some more Proof Constructs: PICK

When Γ contains a statement of the form:

$$\exists x \in S : \; P(x)$$

To use it we can write:

$$\langle 1 \rangle 6 \;\; \text{PICK} \;\; a \in S : P(a)$$

for some fresh a.

The effect is:

1. we have a new $a \in S$ in Γ;
2. using $\langle 1 \rangle 6$ will put $P(a)$ into Γ.

SUFFICES is useful to avoid deeply nested hierarchical proofs:

$\langle 6 \rangle 4$ X
  $\langle 7 \rangle$ proof Π

$\langle 6 \rangle q$ QED
  BY $\langle 6 \rangle 4$, proof Σ

$\langle 6 \rangle 4$ SUFFICES X
  proof Σ

$\langle 6 \rangle 5$ proof Π

$\langle 6 \rangle q$ QED
  BY $\langle 6 \rangle 4$, $\langle 6 \rangle 5$

Outline

A simple theorem about sequences

- *Concat left cancellation:* Given three sequences A, B, C where $C \circ A = C \circ B$, it follows that $A = B$.
  - Simple, but not trivial. Multiplication, for example, does not have left cancellation, because you can multiply by zero.

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  As a quantified formula:

  $$\forall S : \forall A, B, C \in Seq(S) :$$
  $$C \circ A = C \circ B \Rightarrow A = B$$

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As a quantified formula:

$$\forall\, S : \forall\, A, B, C \in Seq(S) : \\ C \circ A = C \circ B \Rightarrow A = B$$

As an ASSUME-PROVE:

$$\begin{array}{l} \text{ASSUME} \\ \quad \text{NEW } S, \\ \quad \text{NEW } A \in Seq(S), \\ \quad \text{NEW } B \in Seq(S), \\ \quad \text{NEW } C \in Seq(S), \\ \quad C \circ A = C \circ B \\ \text{PROVE } A = B \end{array}$$

Proof attempt 1 - is it obvious - fail

THEOREM $ConcatLeftCancel \triangleq$
  ASSUME
    NEW S,
    NEW $A \in Seq(S)$,
    NEW $B \in Seq(S)$,
    NEW $C \in Seq(S)$,
    $C \circ A = C \circ B$
  PROVE
  $A = B$
PROOF
  $\langle 1 \rangle$ QED OBVIOUS   unable to prove it

Proof attempt 2 - add some facts - still fail

THEOREM $ConcatLeftCancel \triangleq$
  ASSUME
    NEW S,
    NEW $A \in Seq(S)$,
    NEW $B \in Seq(S)$,
    NEW $C \in Seq(S)$,
    $C \circ A = C \circ B$
  PROVE
  $A = B$
PROOF
  $\langle 1 \rangle 1.\ Len(A) = Len(B)$ OBVIOUS $C \circ A = C \circ B$
  $\langle 1 \rangle 2.\ A \in [1 \,..\, Len(A) \rightarrow S]$ OBVIOUS $A \in Seq(S)$
  $\langle 1 \rangle 3.\ B \in [1 \,..\, Len(A) \rightarrow S]$ BY $\langle 1 \rangle 1$
  $\langle 1 \rangle 4.\ \forall\, i \in 1 \,..\, Len(A) : A[i] = (C \circ A)[i + Len(C)]$ OBVIOUS
  $\langle 1 \rangle 5.\ \forall\, i \in 1 \,..\, Len(A) : B[i] = (C \circ B)[i + Len(C)]$ BY $\langle 1 \rangle 1$
  $\langle 1 \rangle$ QED BY $\langle 1 \rangle 2, \langle 1 \rangle 3, \langle 1 \rangle 4, \langle 1 \rangle 5$ unable to prove it

Proof attempt 3 - another fact - success

THEOREM $ConcatLeftCancel \triangleq$
  ASSUME
    NEW S,
    NEW $A \in Seq(S)$,
    NEW $B \in Seq(S)$,
    NEW $C \in Seq(S)$,
    $C \circ A = C \circ B$
  PROVE
  $A = B$
PROOF
  $\langle 1 \rangle 1.\ Len(A) = Len(B)$ OBVIOUS $C \circ A = C \circ B$
  $\langle 1 \rangle 2.\ A \in [1 \,..\, Len(A) \rightarrow S]$ OBVIOUS $A \in Seq(S)$
  $\langle 1 \rangle 3.\ B \in [1 \,..\, Len(A) \rightarrow S]$ BY $\langle 1 \rangle 1$
  $\langle 1 \rangle 4.\ \forall i \in 1 \,..\, Len(A) : A[i] = (C \circ A)[i + Len(C)]$ OBVIOUS
  $\langle 1 \rangle 5.\ \forall i \in 1 \,..\, Len(A) : B[i] = (C \circ B)[i + Len(C)]$ BY $\langle 1 \rangle 1$
  $\langle 1 \rangle 6.\ \forall i \in 1 \,..\, Len(A) : A[i] = B[i]$ BY $\langle 1 \rangle 4, \langle 1 \rangle 5$
  $\langle 1 \rangle$ QED BY $\langle 1 \rangle 2, \langle 1 \rangle 3, \langle 1 \rangle 6$

Proof with better structure

THEOREM $ConcatLeftCancel \triangleq$
  ASSUME
    NEW S,
    NEW $A \in Seq(S)$,
    NEW $B \in Seq(S)$,
    NEW $C \in Seq(S)$,
    $C \circ A = C \circ B$
  PROVE
  $A = B$
PROOF
  $\langle 1 \rangle 1.$ $Len(A) = Len(B)$ OBVIOUS $C \circ A = C \circ B$
  $\langle 1 \rangle 2.$ $A \in [1 \,..\, Len(A) \rightarrow S]$ OBVIOUS $A \in Seq(S)$
  $\langle 1 \rangle 3.$ $B \in [1 \,..\, Len(A) \rightarrow S]$ BY $\langle 1 \rangle 1$
  $\langle 1 \rangle 4.$ ASSUME NEW $i \in 1 \,..\, Len(A)$ PROVE $A[i] = B[i]$
    $\langle 2 \rangle 1.$ $A[i] = (C \circ A)[i + Len(C)]$ OBVIOUS   defn of $C \circ A$
    $\langle 2 \rangle 2.$ $B[i] = (C \circ B)[i + Len(C)]$ BY $\langle 1 \rangle 1$ defn of $C \circ B$
    $\langle 2 \rangle$ QED BY $\langle 2 \rangle 1$, $\langle 2 \rangle 2$
  $\langle 1 \rangle$ QED BY $\langle 1 \rangle 2$, $\langle 1 \rangle 3$, $\langle 1 \rangle 4$

Lessons from proving *ConcatLeftCancel*

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- The proof centers on showing $A = B$ where A, B are functions
  - For two functions to be equal, you must show
    - they have the same domain
    - they have the same value at each point in the domain
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- Before writing a subproof, check if TLAPS thinks a fact is obvious
- When TLAPS fails, try to figure out what specific fact you could provide that it is failing to consider
- When introducing a new symbol x, generally it is a good idea to use a domain formula $x \in S$

Using the theorem *ConcatLeftCancel*

Often, what a theorem considers as constant parameters are messy formulas at the point where we wish to apply the theorem. In this example, we conjure up formulas that happen to be sequences, and ask TLAPS to apply *ConcatLeftCancel*.

Use attempt 1 - is it obvious - fail

THEOREM $UseConcatLeftCancel \triangleq$
  ASSUME
    NEW S,
    NEW $u \in Seq(S)$,
    NEW $v \in Seq(S)$,
    NEW $w \in Seq(S)$,
    NEW $x \in Seq(S)$,
    NEW $m \in S$,
    NEW $n \in S$,
    $u \circ \langle m, n \rangle \circ v \circ x = u \circ \langle m, n \rangle \circ w \circ x$
  PROVE
  $v \circ x = w \circ x$
PROOF
  $\langle 1 \rangle$ QED BY $ConcatLeftCancel$ unable to prove it

Use attempt 2 - add a closure fact - still fail

THEOREM $UseConcatLeftCancel \triangleq$
  ASSUME
    NEW S,
    NEW $u \in Seq(S)$,
    NEW $v \in Seq(S)$,
    NEW $w \in Seq(S)$,
    NEW $x \in Seq(S)$,
    NEW $m \in S$,
    NEW $n \in S$,
    $u \circ \langle m, n \rangle \circ v \circ x = u \circ \langle m, n \rangle \circ w \circ x$
  PROVE
  $v \circ x = w \circ x$
PROOF
  $\langle 1 \rangle 1.\ u \circ \langle m, n \rangle \in Seq(S)$ OBVIOUS ◦ closed
  $\langle 1 \rangle$ QED BY $\langle 1 \rangle 1$, $ConcatLeftCancel$ unable to prove it

Use attempt 3 - add more closure facts - still fail

THEOREM $UseConcatLeftCancel \triangleq$
  ASSUME
    NEW S,
    NEW $u \in Seq(S)$,
    NEW $v \in Seq(S)$,
    NEW $w \in Seq(S)$,
    NEW $x \in Seq(S)$,
    NEW $m \in S$,
    NEW $n \in S$,
    $u \circ \langle m, n \rangle \circ v \circ x = u \circ \langle m, n \rangle \circ w \circ x$
  PROVE
    $v \circ x = w \circ x$
PROOF
  $\langle 1 \rangle 1.\ u \circ \langle m, n \rangle \in Seq(S)$ OBVIOUS ◦ closed
  $\langle 1 \rangle 2.\ v \circ x \in Seq(S)$ OBVIOUS ◦ closed
  $\langle 1 \rangle 3.\ w \circ x \in Seq(S)$ OBVIOUS ◦ closed
  $\langle 1 \rangle$ QED BY $\langle 1 \rangle 1$, $\langle 1 \rangle 2$, $\langle 1 \rangle 3$, $ConcatLeftCancel$ unable to prove it

Use attempt 4 - add an associativity fact - success

THEOREM $UseConcatLeftCancel \triangleq$
  ASSUME
    NEW S,
    NEW $u \in Seq(S)$,
    NEW $v \in Seq(S)$,
    NEW $w \in Seq(S)$,
    NEW $x \in Seq(S)$,
    NEW $m \in S$,
    NEW $n \in S$,
    $u \circ \langle m, n \rangle \circ v \circ x = u \circ \langle m, n \rangle \circ w \circ x$
  PROVE
  $v \circ x = w \circ x$
PROOF
  $\langle 1 \rangle 1.\ u \circ \langle m, n \rangle \in Seq(S)$ OBVIOUS ○ closed
  $\langle 1 \rangle 2.\ v \circ x \in Seq(S)$ OBVIOUS ○ closed
  $\langle 1 \rangle 3.\ w \circ x \in Seq(S)$ OBVIOUS ○ closed
  $\langle 1 \rangle 4.\ u \circ \langle m, n \rangle \circ (v \circ x) = u \circ \langle m, n \rangle \circ (w \circ x)$ OBVIOUS
  $\langle 1 \rangle$ QED BY $\langle 1 \rangle 1, \langle 1 \rangle 2, \langle 1 \rangle 3, \langle 1 \rangle 4, ConcatLeftCancel$

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Finite induction over naturals

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- TLAPS has a libary theorem *NatInduction*, in the library module *NaturalsInduction*, that encapsulates the simple inductive argument. For any $P(\_)$, given the base case

$$P(0)$$

and the inductive step

$$\forall i \in Nat : P(i) \Rightarrow P(i+1)$$

NatInduction concludes

$$\forall i \in Nat : P(i)$$

Finite induction over naturals - 2

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- In such cases, we could, of course, define a more general predicate

$$Q(i) \triangleq i \in m..n \Rightarrow P(i)$$

use *NatInduction* to prove that $Q(i)$ holds for all $i \in Nat$ and then deduce what we want about $P(\_)$. But the proof would be cluttered with the transitions of i into and out of $m..n$.

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- In such cases, we could, of course, define a more general predicate

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  use *NatInduction* to prove that $Q(i)$ holds for all $i \in Nat$ and then deduce what we want about $P(\_)$. But the proof would be cluttered with the transitions of i into and out of $m..n$.
- A better approach is to define a prove and prove a theorem *FiniteNatInduction* that explicitly deals with finite induction over the naturals.

Setting up the inductive argument

THEOREM $FiniteNatInduction \triangleq$
  ASSUME
    NEW $P(\_)$, predicate
    NEW $m \in Nat$, start
    NEW $n \in Nat$, limit
    $P(m)$, base case
    $\forall\, i \in m \,..\, (n-1) : P(i) \Rightarrow P(i+1)$ finite ind hyp
  PROVE $\forall\, i \in m \,..\, n : P(i)$
PROOF
  $\langle 1 \rangle$ DEFINE $Q(i) \triangleq i \in m \,..\, n \Rightarrow P(i)$
  $\langle 1 \rangle$ SUFFICES $\forall\, i \in Nat : Q(i)$ OBVIOUS
  base case
  $\langle 1 \rangle 1.\ Q(0)$ OBVIOUS
  inductive step
  $\langle 1 \rangle 2.\ \forall\, i \in Nat : Q(i) \Rightarrow Q(i+1)$
  $\langle 1 \rangle$ HIDE DEF Q hide defn of induction predicate
  $\langle 1 \rangle$ QED BY $\langle 1 \rangle 1$, $\langle 1 \rangle 2$, $NatInduction$

- Define the more general predicate $Q(\_)$
- Use a SUFFICES to change the goal to $\forall\, i \in Nat : Q(i)$
- State the base case and inductive step as facts
- Hide the definition of the inductive predicate $Q(\_)$
- Appeal to $NatInduction$

Completing the subproof of the inductive step

```
THEOREM FiniteNatInduction ≜
  ASSUME
    NEW P(_),              predicate
    NEW m ∈ Nat,           start
    NEW n ∈ Nat,           limit
    P(m),                  base case
    ∀ i ∈ m..(n − 1) : P(i) ⇒ P(i + 1)   finite ind hyp
  PROVE ∀ i ∈ m..n : P(i)
PROOF
  ⟨1⟩ DEFINE Q(i) ≜ i ∈ m..n ⇒ P(i)
  ⟨1⟩ SUFFICES ∀ i ∈ Nat : Q(i) OBVIOUS
  base case
  ⟨1⟩1. Q(0) OBVIOUS
  inductive step
  ⟨1⟩2. ∀ i ∈ Nat : Q(i) ⇒ Q(i + 1)
    ⟨2⟩1. SUFFICES ASSUME NEW i ∈ Nat, Q(i)
             PROVE Q(i + 1) OBVIOUS
    ⟨2⟩2.CASE i + 1 ∈ (m + 1)..n BY ⟨2⟩1, ⟨2⟩2
    ⟨2⟩3.CASE i + 1 = m BY ⟨2⟩3
    ⟨2⟩4.CASE i + 1 ∉ m..n BY ⟨2⟩4
    ⟨2⟩ QED BY ⟨2⟩2, ⟨2⟩3, ⟨2⟩4
  ⟨1⟩ HIDE DEF Q        hide defn of induction predicate
  ⟨1⟩ QED BY ⟨1⟩1, ⟨1⟩2, NatInduction
```

- Use SUFFICES ASSUME PROVE to disassemble the universal quantifier and the implication
- Use CASE to perform a case analysis
- The cases must cover all possibilities

Simplified proof of *FiniteNatInduction*

```
THEOREM FiniteNatInduction ≜
  ASSUME
    NEW P(_),                         predicate
    NEW m ∈ Nat,                      start
    NEW n ∈ Nat,                      limit
    P(m),                             base case
    ∀ i ∈ m .. (n − 1) : P(i) ⇒ P(i + 1)   finite ind hyp
  PROVE ∀ i ∈ m .. n : P(i)
PROOF
  ⟨1⟩ DEFINE Q(i) ≜ i ∈ m .. n ⇒ P(i)
  ⟨1⟩ SUFFICES ∀ i ∈ Nat : Q(i) OBVIOUS
  base case
  ⟨1⟩1. Q(0) OBVIOUS
  inductive step
  ⟨1⟩2. ∀ i ∈ Nat : Q(i) ⇒ Q(i + 1) OBVIOUS
  ⟨1⟩ HIDE DEF Q          hide defn of induction predicate
  ⟨1⟩ QED BY ⟨1⟩1, ⟨1⟩2, NatInduction
```

- It turns out that TLAPS thinks that the inductive step is obvious. We neglected to check this before plunging into the case analysis. Hence the proof can be simplified considerably.

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- Use SUFFICES to change the goal

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- Use SUFFICES to change the goal
- Use SUFFICES ASSUME PROVE to disassemble universal quantifiers and implications

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- Use SUFFICES to change the goal
- Use SUFFICES ASSUME PROVE to disassemble universal quantifiers and implications
- Use CASE statements to disassemble the current goal into cases
  - TLAPS will have to be convinced that all cases are covered
  - Often it can figure this out on its own, but sometimes you need to present the fact explicitly

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- Use SUFFICES ASSUME PROVE to disassemble universal quantifiers and implications
- Use CASE statements to disassemble the current goal into cases
  - TLAPS will have to be convinced that all cases are covered
  - Often it can figure this out on its own, but sometimes you need to present the fact explicitly
- Always check to see if TLAPS can prove a fact (given the necessary predicate facts) before plunging into a subproof

Outline

Temporal proofs in TLAPS

```
LEMMA TypeOK_inv == Spec => []TypeOK
<1>1. Init => TypeOK
  BY NAssumption DEF Init, TypeOK, Nodes, Color
<1>2. TypeOK /\ [Next]_vars => TypeOK'
  <2>. USE DEF Nodes, Color
  <2>. SUFFICES ASSUME TypeOK, Next \/ UNCHANGED vars
                PROVE  TypeOK'
    OBVIOUS
  <2>1. CASE InitiateProbe
    BY <2>1, NAssumption DEF InitiateProbe, TypeOK
  <2>2. ASSUME NEW i \in Nodes \ {0}, PassToken(i)
        PROVE  TypeOK'
    BY <2>2, NAssumption DEF PassToken, TypeOK
  <2>3. ASSUME NEW i \in Nodes, SendMsg(i)
        PROVE  TypeOK'
    BY <2>3, NAssumption DEF SendMsg, TypeOK
  <2>4. ASSUME NEW i \in Nodes, Deactivate(i)
        PROVE  TypeOK'
    BY <2>4 DEF Deactivate, TypeOK
  <2>5. CASE UNCHANGED vars
    BY <2>5 DEF vars, TypeOK
  <2>. QED  BY <2>1, <2>2, <2>3, <2>4, <2>5 DEF Next
<1>3. QED
  BY <1>1, <1>2, PTL DEF Spec
```

- A standard safety proof
  - validation of a temporal formula
  - mostly action reasoning
  - temporal reasoning for validating QED step

Temporal proofs in TLAPS

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- A standard safety proof
  - validation of a temporal formula
  - mostly action reasoning
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- In this talk:
  - Why

    quantified temporal formulas can be proved using

    first-order and propositional temporal backends

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```

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  - validation of a temporal formula
  - mostly action reasoning
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  - How

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Temporal proofs in TLAPS

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- A standard safety proof
  - validation of a temporal formula
  - mostly action reasoning
  - temporal reasoning for validating QED step
- In this talk:
  - Why

    quantified temporal formulas can be proved using

    first-order and propositional temporal backends
  - How

    to write the proofs correctly
  - Which

    formulas can be proved using that
    - Note: TLA^+ is not complete for quantified temporal logic.

Temporal concepts in TLA$^+$

```
(*********
--algorithm Simple {
   variables x = 0; {
   while (TRUE) {
    x := x + 3;
   }}
}
********)
\* BEGIN TRANSLATION
VARIABLE x
vars == << x >>
Init == x = 0
Next == x' = x + 3
Spec == Init /\ [][Next]_vars
\* END TRANSLATION
------------------------
P == x >= 0
THEOREM Spec => []P
```

Temporal concepts in TLA$^+$

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  - Program Executions

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- Semantics
  - Program Executions
  - States

Temporal concepts in TLA$^+$

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- Semantics
  - Program Executions
  - States
  - Behaviors and suffixes

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- Semantics
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  - Behaviors and suffixes
- Syntax
  - Constant expressions

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- Syntax
  - Constant expressions
  - State expressions

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- Semantics
  - Program Executions
  - States
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- Syntax
  - Constant expressions
  - State expressions
  - Action expressions

Temporal concepts in TLA$^+$

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- Semantics
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- Semantics
  - Program Executions
  - States
  - Behaviors and suffixes
- Syntax
  - Constant expressions
  - State expressions
  - Action expressions
  - Temporal expressions
- Logic
  - First-order [1]

[1] Coalescing: Syntactic Abstraction for Reasoning in First-Order Modal Logics

Temporal concepts in TLA$^+$

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  - Constant expressions
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- Logic
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  - Temporal

[1] Coalescing: Syntactic Abstraction for Reasoning in First-Order Modal Logics

Temporal concepts in TLA+

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   }}
}
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vars == << x >>
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Next == x' = x + 3
Spec == Init /\ [][Next]_vars
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- Syntax
  - Constant expressions
  - State expressions
  - Action expressions
  - Temporal expressions
- Logic
  - First-order [1]
  - Temporal
    - PTL

[1] Coalescing: Syntactic Abstraction for Reasoning in First-Order Modal Logics

Breaking temporal formulas into action formulas

- Proving quantified temporal formulas from action formulas and propositional temporal rules.

Breaking temporal formulas into action formulas

- Proving quantified temporal formulas from action formulas and propositional temporal rules.
  - find a temporal rule

Breaking temporal formulas into action formulas

- Proving quantified temporal formulas from action formulas and propositional temporal rules.
  - find a temporal rule
  - verify the rule

Breaking temporal formulas into action formulas

- Proving quantified temporal formulas from action formulas and propositional temporal rules.
  - find a temporal rule
  - verify the rule
  - understand failures

How to find the rules

- Safety properties - based on variations of the inductive invariant rule:

```
THEOREM Inductive_Invariant ==
    ASSUME STATE I, STATE V,
           ACTION N, STATE P,
           I => P,
           P /\ [N]_V => P'
    PROVE I /\ [][N]_V => []P
```

How to find the rules

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```

```
LEMMA TypeOK_inv == Spec => []TypeOK
<1>1. Init => TypeOK
<1>2. TypeOK /\ [Next]_vars => TypeOK'
<1>3. QED
  BY <1>1, <1>2, PTL DEF Spec
```

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  BY <1>1, <1>2, PTL DEF Spec
```

```
THEOREM Spec => []MutualExclusion
<1>1. Init => Inv
<1>2. Inv /\ [Next]_vars => Inv'
<1>3. Inv => MutualExclusion
<1>4. QED
  BY <1>1, <1>2, <1>3, PTL DEF Spec
```

How to find the rules

- Safety properties - based on variations of the inductive invariant rule:

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    ASSUME STATE I, STATE V,
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```

```
THEOREM Spec => []StructOK1
<1>. USE DEFS Ballot, TypeOK, StructOK1
<1>1. Init => StructOK1
<1>2. TypeOK /\ StructOK1 /\ [Next]_vars => StructOK1'
<1>q. QED
  BY ONLY <1>1, <1>2, typing, PTL DEF Spec
```

How to find the rules

- Safety properties - based on variations of the inductive invariant rule:

```
THEOREM Inductive_Invariant ==
    ASSUME STATE I, STATE V,
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    PROVE I /\ [][N]_V => []P
```

- Other properties - other rules

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<1>1. Init => StructOK1
<1>2. TypeOK /\ StructOK1 /\ [Next]_vars => StructOK1'
<1>q. QED
  BY ONLY <1>1, <1>2, typing, PTL DEF Spec
```

Are the rules sound?

```
LEMMA TypeOK_inv == Spec => []TypeOK
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- Rule is an instance of the PTL rule:

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```

- Success of PTL backend verifies this

Understanding failures

```
LEMMA TypeOK_inv == ASSUME Spec PROVE []TypeOK
<1>1. Init => TypeOK
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```

- Consider this valid lemma

Understanding failures

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```

- Consider this valid lemma
- which seems to be an instance of the PTL rule:

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   PROVE  []P
```

Understanding failures

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LEMMA TypeOK_inv == ASSUME Spec PROVE []TypeOK
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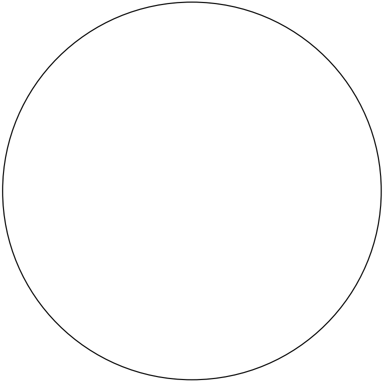
         P /\ [N]_V => P'
  PROVE  []P
```

- But it is not, why?

Necessitation



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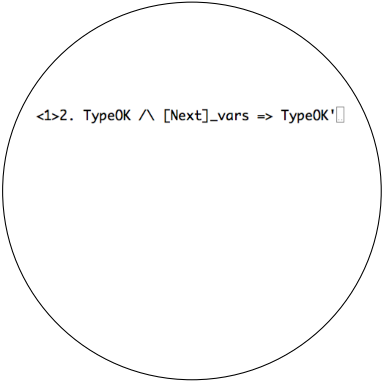


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  BY <1>1, <1>2, PTL DEF Spec
```

Necessitation

```
<1>2. TypeOK /\ [Next]_vars => TypeOK'

<1>2. [](TypeOK /\ [Next]_vars => TypeOK')
```

```
LEMMA TypeOK_inv == Spec => []TypeOK
<1>1. Init => TypeOK
<1>2. TypeOK /\ [Next]_vars => TypeOK'
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- Since $\langle 1 \rangle 2$ holds in all behaviours, it can be boxed
- This is called necessitation

Necessitation

```
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<1>2. [](TypeOK /\ [Next]_vars => TypeOK')
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```
LEMMA TypeOK_inv == Spec => []TypeOK
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<1>3. QED
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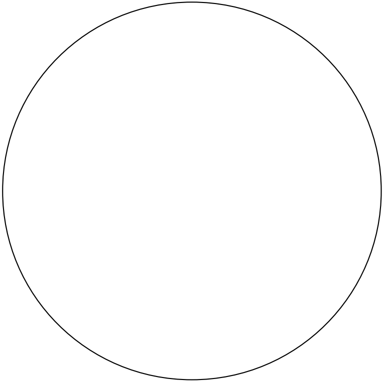
```

- Since $\langle 1 \rangle 2$ holds in all behaviours, it can be boxed
- This is called necessitation
- The PTL rules normally requires the application of necessitation on the action steps

Necessitation



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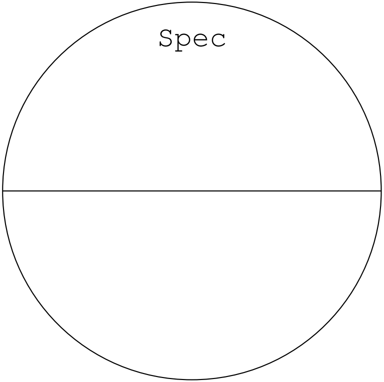


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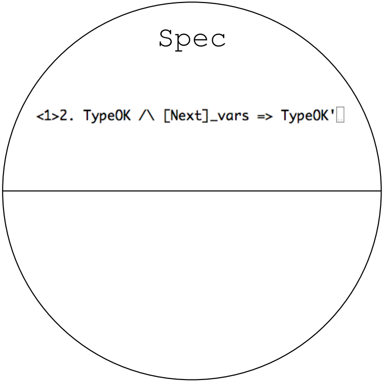
```

- Spec is assumed when proving the proof steps

Necessitation

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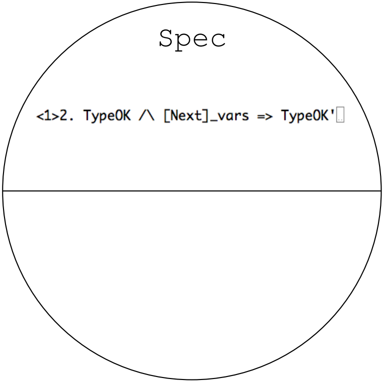
- Spec is assumed when proving the proof steps
- $\langle 1 \rangle 2$ doesn't hold in all behaviours

Necessitation

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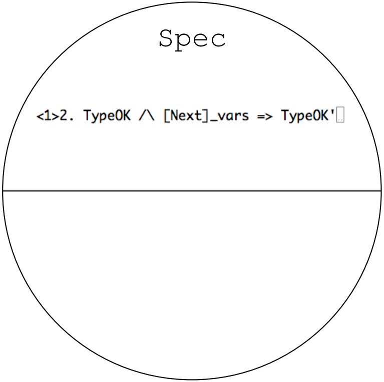
```

- Spec is assumed when proving the proof steps
- $\langle 1 \rangle 2$ doesn't hold in all behaviours
- Necessitation is not applied

Necessitation

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  BY <1>1, <1>2, PTL DEF Spec
```

- Spec is assumed when proving the proof steps
- $\langle 1 \rangle 2$ doesn't hold in all behaviours
- Necessitation is not applied
- Note: There is a workaround

Necessitation and assumptions

```
VARIABLE x

THEOREM ASSUME x=0 PROVE [][x'=x+1]_x => [](x \in {0,1})
<1>1. x=0 => x \in {0,1}
  OBVIOUS
<1>2. x\in {0,1} /\ x'=x+1 => x' \in {0,1}
  OBVIOUS
<1>3. QED BY <1>1,<1>2,PTL
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- Consider the following clearly invalid claim

Necessitation and assumptions

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- The two hypothesis are valid but the rule is not sound
- Why? Necessitation fails for $\langle 1 \rangle 2$

Necessitation and assumptions

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Obligation 1 – status : ls4 : failed

Stop Proving | Goto Obligation

```
ASSUME NEW VARIABLE x,
       x = 0 (* non-□ *),
       x = 0 => x \in {0, 1} (* non-□ *),
       x \in {0, 1} /\ x' = x + 1 => x' \in {0, 1} (* non-□ *)
PROVE  □[x' = x + 1]_x => □(x \in {0, 1})
```

- Consider the following clearly invalid claim
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- Confusing? Necessitation failures are reported in the obligation window

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- Now, the claim is valid, even if in a trivial way
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- What is the difference?

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- Assumptions P, such that $P \Leftrightarrow \Box P$, allow for necessitation.

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» is_?(F) -> True if F is constant
» is_?(True | False) -> True
» is_box([]F) -> True
» is_diamond(<>F) -> True
» is_?([]F | <>F | F') -> is_?(F)
» is_box(~F) -> is_diamond(F)
» is_diamond(~F) -> is_box(F)
» is_?(F /\ G | F \/ G) -> is_?(F) /\ is_?(G) and similarly for lists
» is_box(F -> G) -> is_diamond(F) /\ is_box(G) and similarly for {{is_diamond(F -> G)} |
  is_?(F <-> G)}
» is_?(op(a,b,...)) -> is_?(a) /\ is_?(b) /\ ...
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- An assumption proved in the scope of a non-boxed assumption is considered as non-boxed as well

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- Which:
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    - some proofs require full quantified temporal reasoning - Ex: $\forall x.\Box P(x) \Leftrightarrow \Box\forall x.P(x)$